

ABSOLUTELY CONTINUOUS SPECTRUM OF A SCHRÖDINGER OPERATOR ON A TREE

S. KUPIN

ABSTRACT. We give sufficient conditions for the presence of the absolutely continuous spectrum of a Schrödinger operator on a regular rooted tree without loops (also called regular Bethe lattice or Cayley tree).

INTRODUCTION AND RESULTS

The spectral properties of Schrödinger operators on graphs have numerous applications in physics and they have been intensively studied since late 90's.

We will be mainly interested in the properties of the absolutely continuous component of the spectral measure of a discrete Schrödinger operator H_V on a tree, see (0.2) for an example. Probably, the first specific results in this direction were obtained by Klein [11] who proved the presence of the absolutely continuous component for H_V 's with random iid potential on a regular Bethe lattice. Recently, Aizenman-Sims-Warzel [1] obtained the result with the help of a new general method. They also handled quasi-periodic operators [3] and a Laplacian on a random quantum tree [2]; see Aizenman-Sims-Warzel [4] for a nice overview of the topic. We also mention interesting papers by Froese-Hasler-Spitzer [8, 9] and Breuer [6].

Almost simultaneously to the above-mentioned works, Killip-Simon [10], Nazarov-Peherstorfer-Volberg-Yuditskii [15] obtained important results in the spectral theory of one-dimensional (1D) Schrödinger operators and, more generally, Jacobi matrices. These and subsequent papers [12, 13, 14, 18, 19] gave a fairly complete picture of the spectral behavior of these 1D objects.

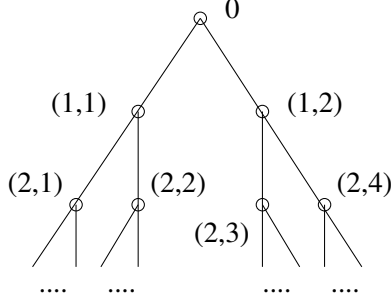
It was hence very tempting to apply the well-developed methods of the one-dimensional analysis to the spectral problems for Schrödinger operators on trees. The first step in this direction was made by Denisov [7], who succeeded to carry over methods of Simon [16] to H_V described in (0.2).

However, the general picture remained quite unclear. In particular, we did not understand to what extent the construction for 1D worked for Schrödinger operators on trees. This gap is fixed by the present paper. Amongst other results, we prove the Main Lemma (see Section 1) which expresses the Jost solutions for H_V in terms of corresponding perturbation determinants. This observation implies immediately that there are strong parallels between spectral behavior of 1D Schrödinger operators and similar objects on trees, and we recover a big part of the 1D theory for these H_V 's. In particular, the sum rules of higher order for 1D Jacobi matrices

Date: May, 15, 2008.

1991 Mathematics Subject Classification. Primary: 34L40.

Key words and phrases. Absolutely continuous spectrum, Schrödinger operator, Cayley tree, Bethe lattice.

FIGURE 1. The tree T

become the “sum inequalities” for H_V . These higher order sum inequalities are proved in Theorems 0.2, 0.3.

Let $T = T_0$ be a regular binary tree of without loops (also called Cayley tree or Bethe lattice). Its root vertex is denoted by 0. The set of all vertices of the tree is denoted by $\mathcal{V}(T)$. The distance $|x_1 - x_2|$ between two vertices $x_1, x_2 \in \mathcal{V}(T)$ is the number of edges of the (unique) path leading from x_1 to x_2 . The sphere of radius n and centered at 0 is

$$\mathcal{S}(0, n) = \{x \in \mathcal{V}(T) : |x| = |x - 0| = n\}.$$

Every vertex in the tree has one ascendant and two descendants. The descendants of different vertices are different since the tree does not have closed loops. So, the sphere $\mathcal{S}(0, n)$ contains 2^n vertices $x = (n, k)$, $k = 1, \dots, 2^n$. That is,

- $\mathcal{V}(T) = \{0, (n, k) : n \in \mathbb{N}, k = 1, \dots, 2^n\}$,
- the descendants of 0 are vertices $(1, 1), (1, 2)$,
- for $n \geq 1$, the descendants of $(n, j), j = 1, \dots, 2^n$, are $(n+1, 2j-1)$ and $(n+1, 2j)$, see Figure 1.

Let

$$(0.1) \quad l^p(T) = \{\{u(x)\}_{x \in \mathcal{V}(T)} : \sum_{x \in \mathcal{V}(T)} |u(x)|^p < \infty\}$$

with $1 \leq p \leq \infty$. The standard “basis” vectors are $\{e_x\}_{x \in \mathcal{V}(T)}$, where $e_x(x) = 1$ and $e_x(y) = 0$ for $y \neq x$, $y \in \mathcal{V}(T)$. The free Laplacian $H_0 = H_{0,T}$ is defined as

$$(H_0 f)(x) = \sum_{x' : |x' - x| = 1} f(x'),$$

where $f \in l^2(T)$. We also set

$$m_{H_0}(z) = ((H_0 - z)^{-1} e_0, e_0) = \int_{\mathbb{R}} \frac{d\mu_0}{x - z}$$

to be the Weyl-Titchmarsh function of the operator. The Borel measure μ_0 is called the spectral measure of H_0 (with respect to e_0). The spectrum $\sigma(H_0)$ coincides with $\text{supp } \mu_0$, and $\sigma(H_0) = \sigma_{ac}(H_0) = [-2\sqrt{2}, 2\sqrt{2}]$, see for example [7, Sect. 2].

A Schrödinger operator on $l^2(T)$ is a diagonal perturbation of H_0 ,

$$(0.2) \quad (H_V f)(x) = \sum_{x': |x'-x|=1} f(x') + V(x).$$

We always assume that $V = \{V(x)\}_{x \in \mathcal{V}(T)}$ lies in $c_0(T)$, where

$$c_0(T) = \{\{u(x)\}_{x \in \mathcal{V}(T)} : \lim_{|x| \rightarrow +\infty} u(x) = 0\}.$$

Then the operator H_V is self-adjoint and, once again, we define its spectral measure $\mu = \mu_{H_V}$ as

$$m_{H_V}(z) = ((H_V - z)^{-1} e_0, e_0) = \int_{\mathbb{R}} \frac{d\mu}{x - z}.$$

Since $H_V - H_0$ is compact, the Weyl-von Neumann theorem says that the essential spectrum $\sigma_{ess}(H_V)$ of the operator H_V equals $[-2\sqrt{2}, 2\sqrt{2}]$, and the point spectrum $\sigma_p(H_V) \subset \mathbb{R} \setminus \sigma_{ess}(H_V)$ accumulates to the points $\pm 2\sqrt{2}$ only. It is convenient to enumerate $\sigma_p(H_V) = \{x_{V,T;s}^{\pm}\}$ as follows

$$(0.3) \quad x_{V,T;1}^- \leq \dots \leq x_{V,T;s}^- \leq \dots < -2\sqrt{2},$$

and

$$(0.4) \quad 2\sqrt{2} < \dots \leq x_{V,T;s}^+ \leq \dots \leq x_{V,T;1}^+.$$

The numbering takes into account the (geometric) multiplicities of the eigenvalues.

For a given potential $V = \{V(x)\}_{x \in \mathcal{V}(T)}$, define its “truncation” as

$$V(n) = \{V(n; x)\}_{x \in \mathcal{V}(T)} = \begin{cases} V(x), & |x| \leq n, \\ 0, & |x| > n. \end{cases}$$

Let $T_x, x \in \mathcal{V}(T)$, be a subtree of T growing from the vertex x . By H_{V,T_x} we mean the Schrödinger operator with potential $V|_{T_x}$, the restriction of the original potential V to T_x . The notation $\sigma_p(H_{V,T_x}) = \{x_{V,T_x;s}^{\pm}\}$ are self-obvious and stay for the point spectrum and eigenvalues of the operator H_{V,T_x} .

We give sufficient conditions for the support $\sigma_{ac}(H_V)$ of the absolutely continuous part of the measure μ_{H_V} to fill in the interval $[-2\sqrt{2}, 2\sqrt{2}]$. For instance, the following theorem is proved in Denisov [7].

Theorem 0.1. *Let H_V be a Schrödinger operator (0.2), $V \in c_0(T)$, and*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x: |x|=n} V(x)^2 < \infty.$$

Then

$$(0.5) \quad \int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) \cdot \sqrt{8 - x^2} dx > -\infty,$$

$$\limsup_n \left(EV_{V(n),T}^{3/2} - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x: |x|=k} EV_{V(n),T_x}^{3/2} \right) < \infty,$$

where, for $p \geq 1$,

$$(0.6) \quad EV_{V(n),T}^p = \sum_s |x_{V(n),T;s}^+ - 2\sqrt{2}|^p + \sum_s |x_{V(n),T;s}^- + 2\sqrt{2}|^p.$$

Above, μ' is the density of the absolutely continuous part of the measure μ . Notice that the expression at the LHS of (0.5) is actually non-negative.

For a given $V \in l^\infty(T)$, define δV as

$$(\delta V)(n, j) = \begin{cases} V(n-1, i) - V(n, 2i), & j = 2i, \\ V(n-1, i) - V(n, 2i-1), & j = 2i-1, \end{cases}$$

We prove the following theorems.

Theorem 0.2. *Let H_V be a Schrödinger operator (0.2), $V \in c_0(T)$, and*

$$(0.7) \quad \begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} V(x)^4 < \infty, \quad \sum_{n=2}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} (\delta V)(x)^2 < \infty, \\ & \limsup_n \left(EV_{V(n),T}^{5/2} - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} EV_{V(n),T_x}^{5/2} \right) < \infty. \end{aligned}$$

Then

$$(0.8) \quad \int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) \cdot (8 - x^2)^{3/2} dx > -\infty.$$

Theorem 0.3. *Let H_V be as in (0.2), $V \in c_0(T)$, and*

$$(0.9) \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} V(x)^6 < \infty, \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} (\delta V)(x)^2 < \infty.$$

Then we have

$$(0.10) \quad \int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) \cdot (8 - x^2)^{5/2} dx > -\infty,$$

$$(0.11) \quad \limsup_n \left(EV_{V(n),T}^{7/2} - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} EV_{V(n),T_x}^{7/2} \right) < \infty.$$

These theorems lead to conjectures stated in Section 2.

Remark 0.4. As in [7], the above theorem can be modified to assert that that relations (0.9) (with $V \in l^\infty(T)$) yield (0.10), and, consequently, $[-2\sqrt{2}, 2\sqrt{2}] \subset \sigma_{ac}(H_V)$. The same applies to Conjecture 2.1.p with odd p 's.

Using results of Borichev-Golinskii-Kupin [5], we can express relations (0.5), (0.7) and (0.11) in terms of $\sigma_p(H_V)$ and the point spectra $\sigma_p(H_{V,T_x})$ of the corresponding operators. Of course, the assumptions on the potential V become considerably more stringent.

Proposition 0.5. *For $p \geq 1$, let $V \in l^{p'}(T)$, $p' < p + 1/2$. Then the limit below exists and*

$$\lim_{n \rightarrow \infty} \left\{ EV_{V(n),T}^{p+1/2} - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} EV_{V(n),T_x}^{p+1/2} \right\} = EV_{V,T}^{p+1/2} - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} EV_{V,T_x}^{p+1/2}.$$

We also have

$$EV_{V,T}^{p+1/2} \leq C(p, p', \|V\|_\infty) \|V\|_{p'}^{p'},$$

where $\|\cdot\|_\infty, \|\cdot\|_p$ are the norms of $l^\infty(T), l^p(T)$, respectively.

1. SKETCH OF THE PROOF OF THEOREM 0.2

As usual, we uniformize the domain $\bar{\mathbb{C}} \setminus [-2\sqrt{2}, 2\sqrt{2}]$ with the help of the maps $z(\zeta) = \sqrt{2}(\zeta + 1/\zeta)$, $\zeta(z) = \frac{1}{2\sqrt{2}}(z - \sqrt{z^2 - 8})$, where $\zeta \in \mathbb{D} = \{\zeta : |\zeta| < 1\}$.

Let for the moment $\text{rank } V < \infty$. We put for an arbitrary subtree $T_x \subset T, x \in \mathcal{V}(T)$,

$$L_{T_x}(\zeta) = L_{T_x}(z(\zeta)) = \det(H_{V,T_x} - z)(H_{0,T_x} - z)^{-1},$$

where H_{0,T_x} is the free Laplacian on T_x . Furthermore, let \tilde{X} be a finite subset of vertices of the tree T with the property $T_x \cap T_y = \emptyset$ for $x, y \in \tilde{X}$ and $x \neq y$. Obviously, we can speak about $H_{T_{\tilde{X}}} = \oplus_{x \in \tilde{X}} H_{T_x}$ and

$$L_{T_{\tilde{X}}}(\zeta) = L_{T_{\tilde{X}}}(z(\zeta)) = \det(H_{V,T_{\tilde{X}}} - z)(H_{0,T_{\tilde{X}}} - z)^{-1} = \prod_{x \in \tilde{X}} L_{T_x}(\zeta).$$

Consider the path γ_y leading from 0 to $y \in \mathcal{V}(T)$ and denote by $\tilde{X}(y)$ the set of vertices lying on the distance one from vertices of the path, that is,

$$\tilde{X}(y) = \{x \in \mathcal{V}(T) : \exists w \in \mathcal{V}(T) \cap \gamma_y, |x - w| = 1\}.$$

It is easy that $\tilde{X}(y)$ has the above-mentioned disjointness property and, moreover, $(\mathcal{V}(T) \cap \gamma_y) \cup \mathcal{V}(T_{\tilde{X}(y)}) = \mathcal{V}(T)$.

The next lemma is the key to the proofs of Theorems 0.1-0.3. It is new and it expresses the Jost solution of the operator H_V in terms of $L_{T_{\tilde{X}(y)}}$, compare to [10, Theorem 2.16]. It goes without saying that the lemma holds also for “sparse” trees considered in [6].

Main Lemma. *Let $\text{rank } V < \infty$ and H_V be the Schrödinger operator (0.2). Let $f(\zeta) = \{f_y(\zeta)\}_{y \in T} \in l^2(T)$ and $f = (H_V - z(\zeta))^{-1}e_0$. Then, for $n = |y|$,*

$$f_y(\zeta) = \left(\frac{\zeta}{\sqrt{2}}\right)^n L_{T_{\tilde{X}(y)}}(\zeta)/L_T(\zeta).$$

The proofs of the theorem use the techniques developed in [10, 15, 12, 13] and the lemma.

Sketch of the proof of Theorem 0.2. Let the potential $V \in c_0(T)$ satisfy the assumptions of the theorem. We do the computations for the operator $H_{V(N)}$, and then pass to the limit with respect to $N \rightarrow \infty$.

Make the change of variables $z(\zeta) = \sqrt{2}(\zeta + \zeta^{-1})$ and transfer the spectral measure $\mu_N = \mu_{H_{V(N)}}$ to the unit disk \mathbb{D} and its boundary. The absolutely continuous part of the image of the measure is then supported on the unit circle and its density is still denoted μ'_N . We write $\{\zeta_{V(N),T_x;s}\}_s$ for the images of $\{x_{V(N),T_x;s}^\pm\}_s$. Then relations (0.7), (0.8) read as

$$\int_0^{2\pi} \log \mu'_N(e^{i\theta}) \sin^4 \theta d\theta > -\infty, \quad \limsup_N \left(F_{V(N),T} - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} F_{V(N),T_x} \right) < \infty,$$

where

$$F_{V(N),T} = \sum_s (1 - |\zeta_{V(N),T_x;s}|)^5.$$

For a given vertex $x \in \mathcal{V}(T)$, we consider $H_{V(N),T_x}$ and the perturbation determinant $L_{T_x}(\zeta)$ ($= \det(H_{V(N),T_x} - z)(H_{0,T_x} - z)^{-1}$). The eigenvalues $\{\zeta_{V(N),T_x;s}\}$ coincide with the zeros of the determinant up to multiplicities. We have in a neighborhood of $\zeta = 0$

$$\log L_{T_x}(\zeta) = - \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} \left(T_k \left(\frac{1}{\sqrt{2}} H_{V(N),T_x} \right) - T_k \left(\frac{1}{\sqrt{2}} H_{0,T_x} \right) \right) \zeta^k,$$

where $T_k(2 \cos \theta) = 2 \cos k\theta$, $k = 0, 1, 2, \dots$, are properly normalized Chebyshev polynomials of the first kind. The following identities hold: for $n = 0$,

$$\frac{1}{4\pi} \int_0^{2\pi} \log |L_{T_x}(e^{i\theta})|^2 d\theta = \sum_s \log 1/|\zeta_{V(N),T_x;s}|,$$

for $n \geq 1$,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log |L_{T_x}(e^{i\theta})|^2 \cos n\theta d\theta &= \frac{1}{n} \sum_s (1/|\zeta_{V(N),T_x;s}|^n - |\zeta_{V(N),T_x;s}|^n) \\ &\quad - \frac{1}{n} \operatorname{tr} \left(T_n \left(\frac{1}{\sqrt{2}} H_{V(N),T_x} \right) - T_n \left(\frac{1}{\sqrt{2}} H_{0,T_x} \right) \right). \end{aligned}$$

Combining these equalities, we get

(1.1)

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log |L_{T_x}(e^{i\theta})|^2 (16 \sin^4 \theta) d\theta &= \sum_s G(|\zeta_{V(N),T_x;s}|) \\ &\quad - \frac{1}{8} \operatorname{tr} \left\{ (H_{V(N),T_x}^4 - 24 H_{V(N),T_x}^2) - (H_{0,T_x}^4 - 24 H_{0,T_x}^2) \right\}, \end{aligned}$$

where

$$G(|\zeta|) = \frac{1}{2} \left\{ \left(\frac{1}{|\zeta|^4} - |\zeta|^4 \right) - 8 \left(\frac{1}{|\zeta|^2} - |\zeta|^2 \right) + 24 \log |\zeta| \right\}.$$

This relation readily implies that

$$\sum_s G(|\zeta_{V(N),T_x;s}|) \asymp \sum_s (1 - |\zeta_{V(N),T_x;s}|)^5.$$

Turning back to the operator $H_{V(N)}$ and its spectral characteristics, we observe that

$$M_{H_{V(N)}}(\zeta) = -m_{H_{V(N)}}(z(\zeta)) = f_0(\zeta),$$

and the computation for $\operatorname{Im} M_{H_{V(N)}}(\zeta)$, $\zeta = e^{i\theta} \in \mathbb{T}$, gives

$$\operatorname{Im} M_{H_{V(N)}}(\zeta) = \sqrt{2} \sin \theta \frac{\sum_{y:|y|=N} |L_{T_{\tilde{X}(y)}}(\zeta)|^2}{2^N |L_T(\zeta)|^2}.$$

The inequality between the arithmetic and the geometric mean ($\frac{1}{n}(a_1 + \dots + a_n) \geq (a_1 a_2 \dots a_n)^{1/n}$ with $a_j \geq 0$) and some simple combinatorics yield

$$\log \frac{\operatorname{Im} M_{H_{V(N)}}(\zeta)}{\sqrt{2} \sin \theta} \geq \frac{1}{2^N} \sum_{y:|y|=N} \log |L_{T_{\tilde{X}(y)}}(\zeta)|^2 - \log |L_T(\zeta)|^2$$

$$= \sum_{j=1}^N \frac{1}{2^j} \sum_{x:|x|=j} \log |L_{T_x}(\zeta)|^2 - \log |L_T(\zeta)|^2.$$

We now apply equality (1.1) to the logarithms in the RHS and transfer the sums corresponding to the point spectra to the LHS of the inequality. So we come to

$$(1.2) \quad \begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\operatorname{Im} M_{H_{V(N)}}(e^{i\theta})}{\sqrt{2} \sin \theta} (16 \sin^4 \theta) d\theta \\ & + \left\{ \sum_s G(|\zeta_{V(N), T_x; s}|) - \sum_{j=1}^N \frac{1}{2^j} \sum_{x:|x|=j} \sum_s G(|\zeta_{V(N), T_x; s}|) \right\} \\ & \geq \frac{1}{8} \operatorname{tr} \left[K(H_{V(N), T}) - K(H_{0, T}) - \sum_{j=1}^N \frac{1}{2^j} \sum_{x:|x|=j} (K(H_{V(N), T_x}) - K(H_{0, T_x})) \right], \end{aligned}$$

where $K(H) = H^4 - 24H^2$. An elementary calculation shows that the expression in the RHS of the relation satisfies the inequality

$$[\dots] \geq -C \left\{ \sum_{j=1}^N \frac{1}{2^j} \sum_{x:|x|=j} V^4(x) + \sum_{j=2}^N \frac{1}{2^j} \sum_{x:|x|=j} (\delta V)(x)^2 \right\}.$$

Now, take \limsup_N of the both sides of inequality (1.2). Its RHS is finite by the assumptions of the theorem. Use the semi-continuity of the entropy (see [10]) to get

$$\limsup_N \int_0^{2\pi} \log \frac{\mu'_N(e^{i\theta})}{\sqrt{2} \sin \theta} \sin^4 \theta d\theta \leq \int_0^{2\pi} \log \frac{\mu'(e^{i\theta})}{\sqrt{2} \sin \theta} \sin^4 \theta d\theta.$$

Hypothesis (0.7) of the theorem says that

$$\limsup_N \left\{ \sum_s G(|\zeta_{V(N), T_x; s}|) - \sum_{j=1}^N \frac{1}{2^j} \sum_{x:|x|=j} \sum_s G(|\zeta_{V(N), T_x; s}|) \right\} < \infty.$$

The proof is complete. \square

2. SOME OPEN QUESTIONS AND CONJECTURES

The following conjecture seems very natural.

Conjecture 2.1.p. *Let H_V be a Schrödinger operator (0.2) on a tree and $V \in c_0(T)$. Let, for an odd $p \geq 1$,*

$$(2.1) \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} V(x)^{2p} < \infty, \quad \sum_{n=2}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} (\delta V)(x)^2 < \infty,$$

Then

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) (8 - x^2)^{p-1/2} dx > -\infty,$$

$$(2.2) \quad \limsup_n \left\{ EV_{V(n),T}^{p+1/2} - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} EV_{V(n),T_x}^{p+1/2} \right\} < \infty.$$

For an even $p \geq 1$, condition (2.2) becomes a hypothesis. The proof of this conjecture should follow the arguments of the theorem obtained in Section 1. The difficulties are mainly of computational character.

The next observation is that the above-mentioned difference derivatives have depend more on the “tree structure” of the operator H_V . Namely, the conjecture is that δV ($\delta^k V$) have to be replaced by $\tilde{\delta} V$ ($\tilde{\delta}^k V$, respectively), where

$$(\tilde{\delta} V)(n, j) = V(n, j) - \frac{1}{2} (V(n+1, 2j-1) + V(n+1, 2j)).$$

The following conjecture contains the previous ones as a very particular case. This is a carry over of Simon’s conjecture [17, Sect. 2.8], supplemented by Nazarov-Peherstorfer-Volberg-Yuditskii [15, Lemma 6.8]. To formulate it, we define an isometry $W = \text{diag} \{A_n\}_n : l^2 \rightarrow l^2(T)$, where $A_n^t = \underbrace{[(1/2)^{n/2}, \dots, (1/2)^{n/2}]}_{2^n \text{ entries}}$.

It is well-known [1], that if V is a radially symmetric potential, that is, $V(x) = \text{const}$ for a fixed $|x|$, then

$$J_{V/\sqrt{2}} = \begin{bmatrix} \frac{V_0}{\sqrt{2}} & 1 & 0 & \dots \\ 1 & \frac{V_1}{\sqrt{2}} & 1 & \dots \\ 0 & 1 & \frac{V_2}{\sqrt{2}} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \frac{1}{\sqrt{2}} W^* H_V W,$$

or $J_{V/\sqrt{2}}$ is unitarily equivalent to a proper restriction of H_V .

We rewrite the conjecture for $J_{V/\sqrt{2}}$ given in [15] in the following way. Let $A(x) = \sum_{k=0}^{2m} a_k x^k$ be a polynomial of degree $2m$ and $A(x) \geq 0$ on \mathbb{R} . Let also

$$(2.3) \quad dH_V = \left[V(0); \underbrace{0, 0}_{2 \text{ entries}}; \underbrace{\frac{1}{2}(V(1, 1), V(1, 2)), \frac{1}{2}(V(1, 1), V(1, 2))}_{2 \times 2 = 4 \text{ entries}}; \right. \\ \left. \underbrace{0, \dots, 0}_{8 \text{ entries}}; \underbrace{\frac{1}{4}(V(2, 1), V(2, 2), V(2, 3), V(2, 4)), \dots}_{4 \times 4 = 16 \text{ entries}} \right]^t.$$

Consider an operator-valued polynomial

$$B(H_0) = \sum_{k=0}^{2m} \frac{a_k}{2^{k/2}} \underbrace{H_0 W \cdot W^* H_0 W \cdot \dots \cdot W^* H_0}_{k \text{ factors}}.$$

Conjecture 2.2. *Let*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} |V(x)|^{2m+2} < \infty, \quad |(B(H_0)dH_V, dH_V)| < \infty.$$

Then

$$\int_{-2\sqrt{2}}^{2\sqrt{2}} \log \mu'(x) A(x/\sqrt{2}) \sqrt{8-x^2} dx > -\infty,$$

$$\limsup_n \left\{ F_{V(n),T}^A - \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{x:|x|=k} F_{V(n),T_x}^A \right\} < \infty,$$

where

$$F_{V(n),T}^A = \sum_s F^A(x_{V(n),T;s}^+) + \sum_s F^A(x_{V(n),T;s}^-),$$

and, for $\pm x^\pm > \pm 2\sqrt{2}$,

$$F^A(x^\pm) = \pm \int_{\pm 2\sqrt{2}}^{x^\pm} A(s/\sqrt{2}) \sqrt{8-s^2} ds.$$

The notation $\{x_{V(n),T;s}^\pm\}_s$ is introduced in (0.3), (0.4).

Several remarks are in order. First, we can formulate a similar conjecture for *Jacobi operators* on trees. The increments of the coefficients associated to the edges of the tree then fill in zero entries in (2.3). Second, consider the usual shift $Se_k = e_{k+1}$ on l^2 , $\{e_k\}_k$ being the standard basis of the space. We now fix the standard basis $\{e_x\}_{x \in \mathcal{V}(T)}$ in $l^2(T)$, see (0.1). The binary shift is defined as

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & \dots \\ B_1 & 0 & 0 & \dots \\ 0 & B_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

where $B_1 = [1, 1]^t$, and B_{k+1} is the $2^{k+1} \times 2^k$ matrix $\begin{bmatrix} B_k & 0 \\ 0 & B_k \end{bmatrix}$. It is plain that $H_0 = S_1 + S_1^*$ and

$$\begin{aligned} S_1 W &= \sqrt{2} W S, & W^* S_1^* &= \sqrt{2} S^* W^*, \\ W^* S_1 &= \sqrt{2} S W^*, & S_1^* W &= \sqrt{2} W S^*. \end{aligned}$$

For instance, we see for Theorem 0.1

$$A(x) = 1, \quad B(H_0) = I, \quad (B(H_0)dH_V, dH_V) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{x:|x|=n} V(x)^2 < \infty,$$

and for Theorem 0.2,

$$\begin{aligned} A(x) &= x^2 - 4, \quad B(H_0) = \frac{1}{2}(H_0 W W^* H_0 - 8) = -\frac{1}{2}(S_1 - S_1^*)^* W W^* (S_1 - S_1^*), \\ (B(H_0)dH_V, dH_V) &= -\sum_{n=1}^{\infty} \frac{1}{2^{n+1}} \sum_{x:|x|=n} \delta V(x)^2 > -\infty. \end{aligned}$$

Roughly speaking, Conjecture 2.2 says that the role of J_0 is played by H_0 and the usual shift S is replaced by S_1 as compared to conjectures [17, Sect. 2.8] and [15, Lemma 6.8]. The presence of the binary shift leads to the “binary” derivatives appearing in the formulations of the theorems.

Acknowledgment. The author would like to thank S. Denisov for helpful discussions.

REFERENCES

- [1] M. Aizenman, R. Sims, S. Warzel, *Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs*. Probab. Theory Related Fields 136 (2006), no. 3, 363–394.
- [2] M. Aizenman, R. Sims, S. Warzel, *Absolutely continuous spectra of quantum tree graphs with weak disorder*. Comm. Math. Phys. 264 (2006), no. 2, 371–389.
- [3] M. Aizenman, R. Sims, S. Warzel, *Fluctuation-based proof of the stability of ac spectra of random operators on tree graphs*. Quantum graphs and their applications, 1–14, Contemp. Math., 415, AMS, Providence, 2006.
- [4] M. Aizenman, S. Warzel, *Persistence under weak disorder of AC spectra of quasi-periodic Schrödinger operators on trees graphs*. Mosc. Math. J. 5 (2005), no. 3, 499–506, 742.
- [5] A. Borichev, L. Golinskii, S. Kupin, *A Blaschke-type condition and its application to complex Jacobi matrices*, submitted.
- [6] J. Breuer, *Singular continuous spectrum for the Laplacian on certain sparse trees*. Comm. Math. Phys. 269 (2007), no. 3, 851–857.
- [7] S. Denisov, *On the preservation of absolutely continuous spectrum for Schrödinger operators*. J. Funct. Anal. 231 (2006), no. 1, 143–156.
- [8] R. Froese, D. Hasler, W. Spitzer, *Transfer matrices, hyperbolic geometry and absolutely continuous spectrum for some discrete Schrödinger operators on graphs*. J. Funct. Anal. 230 (2006), no. 1, 184–221.
- [9] R. Froese, D. Hasler, W. Spitzer, *Absolutely continuous spectrum for the Anderson model on a tree: a geometric proof of Klein’s theorem*. Comm. Math. Phys. 269 (2007), no. 1, 239–257.
- [10] R. Killip, B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*. Ann. of Math. (2) 158 (2003), no. 1, 253–321.
- [11] A. Klein, *Spreading of wave packets in the Anderson model on the Bethe lattice*. Comm. Math. Phys. 177 (1996), no. 3, 755–773.
- [12] S. Kupin, *On sum rules of special form for Jacobi matrices*. C. R. Math. Acad. Sci. Paris 336 (2003), no. 7, 611–614.
- [13] S. Kupin, *On a spectral property of Jacobi matrices*. Proc. Amer. Math. Soc. 132 (2004), no. 5, 1377–1383.
- [14] S. Kupin, *Spectral properties of Jacobi matrices and sum rules of special form*. J. Funct. Anal. 227 (2005), no. 1, 1–29.
- [15] F. Nazarov, F. Peherstorfer, A. Volberg, P. Yuditskii, *On generalized sum rules for Jacobi matrices*. Int. Math. Res. Not. 2005, no. 3, 155–186.
- [16] B. Simon, *A canonical factorization for meromorphic Herglotz functions on the unit disk and sum rules for Jacobi matrices*. J. Funct. Anal. 214 (2004), no. 2, 396–409.
- [17] B. Simon, *Orthogonal polynomials on the unit circle. Part 1*. AMS Colloquium Publications, 54, AMS, Providence, 2005.
- [18] B. Simon, A. Zlotos, *Sum rules and the Szegő condition for orthogonal polynomials on the real line*. Comm. Math. Phys. 242 (2003), 393–423.
- [19] A. Zlotos, *Sum rules for Jacobi matrices and divergent Lieb-Thirring sums*. J. Funct. Anal. 225 (2005), 371–382.

UNIVERSITÉ AIX-MARSEILLE, 39, RUE JOLIOT-CURIE, 13453 MARSEILLE, CEDEX 13, FRANCE

E-mail address: kupin@cmi.univ-mrs.fr